PSEUDOCOMPACT PARATOPOLOGICAL GROUPS THAT ARE TOPOLOGICAL

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Abstract. We obtain necessary and sufficient conditions when a pseudocompact paratopological group is topological. (2-)pseudocompact and countably compact paratopological groups that are not topological are constructed. It is proved that each 2-pseudocompact paratopological group is pseudocompact and that each Hausdorff σ-compact pseudocompact paratopological group is a compact topological group. Our particular attention is devoted to periodic and topologically periodic pseudocompact paratopological groups.

1. Introduction

Let G be a group endowed with a topology τ. A pair (G,τ) is called a *semitopological group* provided the multiplication $\cdot : G\times G\to G$ is separately continuous. Moreover, if the multiplication is continuous then (G,τ) is called a *paratopological group* and the topology τ is called a *semigroup topology* on G. Moreover, if the inversion $(\cdot)^{-1} : G\to G$ is continuous with respect to the topology τ, then (G,τ) is a *topological group*. Properties of paratopological groups, their similarity and difference to properties of topological groups, are described in book [ArhTka] by Alexander V. Arhangel'skii and Mikhail G. Tkachenko, in author's PhD thesis [Rav3] and papers [Rav], [Rav2]. New Mikhail Tkachenko's survey [Tka2] exposes recent advances in this area. A standard example of a paratopological group failing to be a topological group is the Sorgenfrey line, that is the real line endowed with the Sorgenfrey topology (generated by the base consisting of half-intervals $[a,b)$, $a<b$). But it turns out that the inversion on a paratopological group automatically is continuous under some topological conditions. The search and the investigation of these conditions is one of main branches in the theory of paratopological groups and it has a long history. In 1936 Montgomery [Mon] showed that every completely metrizable paratopological group is a topological group. In 1953 Wallace [Wal] asked: *is every locally compact regular semitopological group a topological group?* In 1957 Ellis obtained a positive answer of the Wallace question. In 1960 Zelazko used Montgomery's result and showed that each completely metrizable semitopological group is a topological group. Since both locally compact and completely metrizable topological spaces are Čech-complete (I recall that Čech-complete spaces are G_δ-subspaces of Hausdorff compact spaces), this suggested Pfister [Pfi] in 1985 to ask: *is every Čech-complete semitopological group a topological group?* In 1996 Bouziad [Bou] and Reznichenko [Rez2] independently answered affirmatively to the Pfister's question. To do this, it was sufficient for to show that each Čech-complete semitopological group is a paratopological group since earlier, N. Brand [Bra] had proved that every Čech-complete paratopological group is a topological group. But a problem when a paratopological group is a topological group has many branches which still are growing by efforts of people throughout the world. An interested reader can find known results on this subject in [AlaSan, Introduction], in the survey [Rav3, Section 5.1], in Section 3 of the

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survey [Tka2] and farther in the present paper in which we continue this investigation. We obtain necessary and sufficient conditions when a pseudocompact paratopological group is topological, construct (2-)pseudocompact and countably compact paratopological groups that are not topological, prove that each 2-pseudocompact paratopological group is pseudocompact and that each Hausdorff σ-compact pseudocompact paratopological group is a compact topological group. Our particular attention will be devoted to periodic and topologically periodic pseudocompact paratopological groups. The present paper is a first, slightly updated, part of the authors preprint [Rav7] which grew too large and will be split into several articles.

2. Definitions

Now it is the time and the place to recall the definitions of the used notions.

In this paper the word "space" means "topological space".

Conditions which are close to compactness. In general topology an essential role play compact spaces which have nice properties. These properties are so nice that even spaces satisfying slightly weaker conditions are still useful. We recall that a space X is called

- *sequentially compact* if each sequence of X contains a convergent subsequence,
- *ω-bounded* if each countable subset of X has the compact closure,
- *totally countably compact* if each sequence of X contains a subsequence with the compact closure,
- *countably compact at a subset* A of X if each infinite subset B of A has an accumulation point x in the space X (the latter means that each neighborhood of x contains infinitely many points of the set B),
- *countably compact* if X is countably compact at itself,
- *countably pracompact* if X is countably compact at a dense subset of X,
- *pseudocompact* if each locally finite family of nonempty open subsets of the space X is finite,
- *finally compact* if each open cover of X has a countable subcover.

The following inclusions hold.

- Each compact space is ω-bounded.
- Each ω-bounded space is totally countably compact.
- Each totally countably compact space is countably compact.
- Each sequentially compact space is countably compact.
- Each countably compact space is countably pracompact.
- Each countably pracompact space is pseudocompact.

In these terms, a space X is compact if and only if X is countably compact and finally compact. A Tychonoff space X is pseudocompact if and only if each continuous real-valued function on X is bounded. A paratopological group G is *left (resp. right) precompact* if for each neighborhood U of the unit of G there exists a finite subset F of G such that $FU = G$ (resp. $UF = G$). A paratopological group is left precompact if and only if it is right precompact. So we shall call left precompact paratopological groups *precompact*. A sequence $\{U_n : n \in \omega\}$ of subsets of a space X is *non-increasing* if $U_n \supset U_{n+1}$ for each $n \in \omega$. A paratopological group G is *2-pseudocompact* if $\bigcap \overline{U_n^{-1}} \neq \emptyset$ for each non-increasing sequence $\{U_n : n \in \omega\}$ of nonempty open subsets of G. Clearly, each countably compact paratopological group is 2-pseudocompact. A paratopological group G is *left ω-precompact* if for each neighborhood U of the unit of G there exists a countable subset F of G such that $FU = G$. A paratopological group G is *saturated* if for each nonempty open subset U of G there exists a nonempty open subset V of G such that $V^{-1} \subset U$. Each precompact paratopological group is saturated and each saturated paratopological group is quasiregular.

Separation axioms. All topological spaces considered in the present paper are *not* supposed to satisfy any of the separation axioms, if otherwise is not stated. The definitions and relations between separations axioms both for topological spaces and paratopological groups can be found in the first section of the paper [Rav2]. Each T_0 topological group is $T_{3\frac{1}{2}}$, but for paratopological groups neither of the implications

$$T_0 \Rightarrow T_1 \Rightarrow T_2 \Rightarrow T_3$$

holds (independently of the algebraic structure of the group), Mikhail Tkachenko's survey [Tka2] contains a section devoted to relations between separations axioms for paratopological groups. Some problems about these relations are still open. Paper [Tka2] is devoted to axioms of separation in semitopological groups and related functors. Therefore the separation axioms play essential role in the theory of paratopological groups. In particular, these axioms essentially affect results about automatic continuity of the inversion. We recall some not very well known separation axioms which we shall use in our paper. A space X is T_1 if for every distinct points $x, y \in X$ there exists an open set $x \in U \subset X \backslash \{y\}$. A space X is T_3 if each closed set $F \subset X$ and every point $x \in X \backslash F$ have disjoint neighborhoods. A space X is *regular* if it is T_1 and T_3. A space X is *quasiregular* if each nonempty open subset A of X contains the closure of some nonempty open subset B of X. A space X is a *CB*, if X has a base consisting of canonically open sets, that is such sets U such that $U = \operatorname{int} \overline{U}$. Each T_3 regular space is quasiregular and CB.

Denote by $\mathcal{B}^G$ the open base at the unit of a paratopological group G.

Suppose A is a subset of a group G. Denote by $\langle A \rangle \subset G$ the subgroup generated by the set A.

Now we are ready to go to the next section and to formulate our

3. Results

Proposition 1. *Each totally countably compact paratopological group is a topological group.*

Proof. Let G be such a group. Put $B = \bigcap \mathcal{B}^G$. From Pontrjagin conditions [Rav, 1.1] it follows that B is a semigroup. Put $B' = \overline{\{e\}} = \{x \in G : (\forall U \in \mathcal{B}^G)(xU \ni e)\} = \{x \in G : xB \ni e\} = B^{-1}$. Hence B' is a closed subsemigroup of the group G. Since B' is a compact semigroup, there is a minimal closed right ideal $H \subset B'$. Let x be an arbitrary element of H. Then $x^2 H = H \ni x$. Hence $x^{-1} \in H$ and $e \in H$. Consequently $H = B'$ and $xB' = B'$ for each element $x \in B'$. Therefore B' is a group and $B' = B$. Since $B = \bigcap \mathcal{B}^G$, we see that $g^{-1}Bg \subset B$ for each $g \in G$. Hence B is a normal subgroup of the group G. Since B has the antidiscrete topology, B is a topological group.

Since the set B is closed, we see that the quotient group G/B is a T_1-space. Now let $\{x_n : n \in \omega\}$ be an arbitrary sequence of the group G/B and let $\pi : G \to G/B$ be the quotient map. Choose a sequence $\{x'_n : n \in \omega\}$ of the group G such that $\pi(x'_n) = x_n$ for each $n \in \omega$. Since G is totally countable compact, we see that there is a subsequence A of $\{x'_n : n \in \omega\}$ such that the closure $\overline{A}$ is compact. Since the set $\overline{A}$ is closed, $\overline{A} = \overline{A}B$. Hence the closed compact set $\pi(\overline{A})$ contains a subsequence of the sequence $\{x_n : n \in \omega\}$. Therefore G/B is a T_1 totally countably compact paratopological group. Consequently, from [AlaSan, Cor 2.3] it follows that G/B is a topological group. Hence both B and G/B are topological groups. Then by [Rav4, 1.3] or [Rav3, Pr. 5.3], G is a topological group too. □

Proposition 1 generalizes Corollary 2.3 in [AlaSan] and Lemma 5.4 in [Rav3].

Lemma 1. [Rav2, Pr. 1.7] *Each saturated paratopological group is quasiregular.*

Given a topological space (X, τ) Stone [Sto] and Katetov [Kat] consider the topology τ_r on X generated by the base consisting of all canonically open sets of the space (X, τ). This topology is called the *regularization* of the topology τ. If (X, τ) is a paratopological group then (X, τ_r) is a

T_3 paratopological group [Rav2, Ex. 1.9], [Rav3, p. 31], and [Rav3, p. 28]. We remind that a paratopological group G is *topologically periodic* if for each $x \in G$ and a neighborhood $U \subset G$ of the unit there is a number $n \geq 1$ such that $x^n \in U$, see [BokGur].

Lemma 2. [RavRez] *Suppose (G, τ) is a quasiregular paratopological group such that (G, τ_r) is a topological group; then (G, τ) is a topological group.*

The following lemma is quite easy and probably is known.

Lemma 3. *Let (X, τ) be a topological space. Then (X, τ) is pseudocompact if and only if the regularization (X, τ_r) is pseudocompact.* □

Lemma 4. *Each topologically periodic Baire paratopological group is saturated.*

Proof. In [AlaSan], Alas and Sanchis proved that each T_0 topologically periodic Baire paratopological group is saturated. This result answers an author's Question 2 from [Rav5]. Moreover, from their proof follows that each topologically periodic Baire paratopological group is saturated. □

Lemma 5. *Each 2-pseudocompact paratopological group is a Baire space.*

Proof. In [AlaSan, Th. 2.2], Alas and Sanchis proved that each T_0 2-pseudocompact paratopological group is a Baire space. Nevertheless their proof also proves this lemma. □

Lemma 6. *Each pseudocompact topological group is precompact.* □

Lemma 7. [Rav3, Pr.3.1] *Each precompact paratopological group is saturated.* □

Lemma 8. *(also, see* [AlaSan, Th. 2.5]*) Each Baire left ω-precompact paratopological group is saturated.* □

Lemma 9. *Each paratopological group that is a dense G_δ-subset of a T_3 pseudocompact space is a topological group.*

Proof. In [ArhRez, Th. 1.7], Arhangel'skii and Reznichenko proved that for each paratopological group G such that G is a dense G_δ-subset of a *regular* pseudocompact space, it follows that G is a topological group. Their proof with respective changes also proves this lemma. □

Lemma 10. *Each T_3 2-pseudocompact paratopological group is a topological group.*

Proof. Let G be such a group. Suppose G is not a topological group; then there exists a neighborhood $U \in \mathcal{B}^G$ such that $V \not\subset \overline{U^{-1}} \subset (U^{-1})^2$ for each $V \in \mathcal{B}^G$. By induction we can build a sequence $\{U_i : i \in \omega\}$ of open neighborhoods of the unit e of G such that $\overline{U_0} \subset U$ and $U_{i+1}^2 \subset U_i$ for each $i \in \omega$. Put $F = \bigcap\{U_i : i \in \omega\}$. Then F is a subsemigroup of G. Since the group G is 2-pseudocompact, there is a point $x \in G$ such that $x \in \bigcap \overline{(U_i \backslash \overline{U^{-1}})^{-1}} \subset \bigcap \overline{U_i^{-1}} \subset \bigcap (U_i^{-1})^2 \subset F^{-1}$. Moreover, $x \in \overline{(U_0 \backslash \overline{U^{-1}})^{-1}} \subset \overline{(U_0 \backslash U^{-1})^{-1}} \subset \overline{U_0^{-1} \backslash U} \subset G \backslash U$. Since $x \notin \overline{U_0}$ there is a neighborhood $W \in \mathcal{B}^G$ such that $W \subset U_1$ and $Wx \cap U_0 = \emptyset$. Let $m, n \in \omega$ and $n > m \geq 1$. If $\emptyset \neq Wx^n \cap Wx^m$ then $\emptyset \neq Wx \cap Wx^{m+1-n} \subset Wx \cap U_1 F \subset Wx \cap U_1^2 \subset Wx \cap U_0 = \emptyset$. Therefore $\{Wx^i : i \geq 1\}$ is an infinite family of disjoint sets. Choose a neighborhood $V \in \mathcal{B}^G$ such that $V^2 \subset W$. Since G is 2-pseudocompact, there is a point $y \in \bigcap_{n \geq 1} \overline{(\bigcup_{i \geq n} Vx^i)^{-1}}$. There exist numbers $m, n \in \omega$ such that $n > m \geq 1$ and $yV \cap (Vx^m)^{-1} \neq \emptyset$ and $yV \cap (Vx^n)^{-1} \neq \emptyset$. Then $y^{-1} \in V^2 x^m \cap V^2 x^n \subset Wx^m \cap Wx^n$. This contradiction proves that G is a topological group. □

Each countably compact paratopological group and each pseudocompact topological group are both 2-pseudocompact.

Proposition 2. *Each 2-pseudocompact paratopological group is pseudocompact.*

Proof. Let (G,τ) be such a group. The regularization (G,τ_r) of the group (G,τ) is a T_3 2-pseudocompact paratopological group. By Lemma 10, (G,τ_r) is a topological group. By Lemma 3, (G,τ) is pseudocompact. □

An opposite inclusion does not hold. Manuel Sanchis and Mikhail Tkachenko constructed a Hausdorff pseudocompact Baire not 2-pseudocompact paratopological group [SanTka, Th.2], answering author's questions from a previous version of the manuscript.

Problem 1. *Is each countably pracompact Baire paratopological group 2-pseudocompact?*

The answer to Problem 1 is positive provided the group is quasiregular or saturated or topologically periodic or left ω-precompact.

Proposition 3. *Suppose G is a pseudocompact paratopological group; then the following conditions are equivalent:*

1. *The group G is quasiregular;*
2. *The group G is saturated;*
3. *The group G is topologically periodic Baire;*
4. *The group G is left ω-precompact Baire;*
5. *The group G is precompact;*
6. *The group G is a topological group.*

Proof. It is easy to prove that Condition 6 implies all other conditions.

($1 \Rightarrow 6$) The regularization (G,τ_r) of the group (G,τ) is a T_3 pseudocompact paratopological group. By Lemma 9, (G,τ_r) is a topological group. Then by Lemma 2, (G,τ) is a topological group too.

($2 \Rightarrow 1$) By Lemma 1, the group G is quasiregular.

($3 \Rightarrow 1$) By Lemma 4, the group G is saturated. Then by Lemma 1, the group G is quasiregular.

($4 \Rightarrow 2$) By Lemma 8, the group G is saturated.

($5 \Rightarrow 2$) By Lemma 7, the group G is saturated. □

Implication ($1 \Rightarrow 6$) generalizes Proposition 2 from [RavRez]. Implication ($4 \Rightarrow 6$) generalizes Proposition 2.6 from [AlaSan]. Implication ($3 \Rightarrow 6$) answers Question C from [AlaSan] and generalizes Theorem 3 from [BokGur].

A group has two basic operations: the multiplication and the inversion. The first operation is continuous on a paratopological group, but the second may be not continuous. But there are continuous operations on a paratopological group besides the multiplication. For instance, a power. Now suppose that there exists a non-empty open subset U of a paratopological group G such that the inversion on the set U coincides with a continuous operation. Then, since the inversion on the set U is continuous, it follows that the group G is topological. Let us consider a trivial application of this idea. Let G be a paratopological group of bounded exponent. There is a number n such that the inversion on the group G coincides with the n-th power. Thus the group G is topological. Another applications are formulated in two next propositions.

Proposition 4. *Each Baire periodic paratopological group is a topological group.*

Proof. Let G be such a group. Put $B' = \bigcap\{U^{-1} : U \in \mathcal{B}^G\}$. Then B' is closed subset of G. Since B' is a periodic semigroup, we see that B' is a group. For each $n > 1$ we put $G_n = \{x \in G : x^n \in B'\}$. Since the group G is Baire, there is a number $n > 1$ such that the set G_n has the nonempty interior.

Therefore there are a point $x \in G_n$ and a neighborhood U of the unit of G such that $(xu)^n \in B'$ for each $u \in U$. Then $V^{-1} \subset B'(xV)^{n-1}x$ for each subset V of U. Let U_1 be an arbitrary neighborhood of the unit of G. Then $x^n \in B'U_1$. By the continuity of the multiplication there is a neighborhood $W \subset U$ of the unit such that $(xW)^{n-1}x \subset B'U_1$. Since $B' \subset U_1$, we see that $W^{-1} \subset B'U_1 \subset U_1^2$. □

A Hausdorff topological group G is called *Raikov-complete*, provided it is complete with respect to the upper uniformity which is defined as the least upper bound $\mathcal{L} \vee \mathcal{R}$ of the left and the right uniformities on G. Recall that the sets $\{(x, y) : x^{-1}y \in U\}$, where U runs over a base at unit of G, constitute a base of entourages for the left uniformity $\mathcal{L}$ on G. In the case of the right uniformity $\mathcal{R}$, the condition $x^{-1}y \in U$ is replaced by $yx^{-1} \in U$. The *Raikov completion* $\hat{G}$ of a Hausdorff topological group G is a completion of G with respect to the upper uniformity $\mathcal{L} \vee \mathcal{R}$. For every Hausdorff topological group G the space $\hat{G}$ has a natural structure of a topological group. The group $\hat{G}$ can be defined as a unique (up to an isomorphism) Raikov complete group containing a Hausdorff topological group G as a dense subspace.

Proposition 5. *Each Hausdorff pseudocompact periodic paratopological group is a topological group.*

Proof. Let (G, τ) be such a group and let (G, τ_r) be the regularization of the group (G, τ). Then (G, τ_r) is a regular pseudocompact periodic topological group. Let $\hat{G}$ be the Raikov completion of the group (G, τ_r).

We claim that the group $\hat{G}$ is periodic. Assume the converse. Suppose there exists a non-periodic element $x \in \hat{G}$. Then for each positive integer n there exists a neighborhood $V_n \ni x$ such that $V_n^n \not\ni e$. Since the group G is G_δ-dense in the group $\hat{G}$, we see that there exists a point $y \in G \cap \bigcap V_n$. Then there is a positive integer n such that $y^n = e$. This contradiction proves the periodicity of the group $\hat{G}$.

For each $n > 1$ we put $\hat{G}_n = \{x \in \hat{G} : x^n = e\}$. Since the group $\hat{G}$ is Baire, there is a number $n > 1$ such that the set $\hat{G}_n$ has the nonempty interior. Since the group G is dense in $\hat{G}$, we see that there are a point $x \in G$ and a neighborhood U of the unit of G such that $(xu)^n = e$ for each $u \in U$.

Let y be an arbitrary point of G and V be an arbitrary neighborhood of the point y^{-1}. Then $x^n y^{-1} = y^{-1} \in V$. By the continuity of the multiplication there is a neighborhood $W \in \tau$ of the unit such that $W \subset U$ and $(xW)^{n-1}xy^{-1} \in V$. Let $z \in yW$ be an arbitrary point. Then $(xy^{-1}z)^n = e$. Therefore $z^{-1} = (xy^{-1}z)^{n-1}xy^{-1} \in (xW)^{n-1}xy^{-1} \subset V$. Thus the inversion on the group G is continuous. □

T. Banakh build the next example. This result answers some author's questions and shows that in general, Proposition 5 cannot be generalized for T_1 groups (even for countably pracompact and periodic groups). Moreover, in [AlaSan], Alas and Sanchis proved that each topologically periodic T_0 paratopological group is T_1. Example 1 shows that there is a T_1 periodic paratopological group G such that G is not Hausdorff.

Example 1. *There exists a T_1 periodic paratopological group G such that each power of G is countably pracompact but G is not a topological group.*

Proof. For each positive integer n let C_n be the set $\{0, \ldots, n-1\}$ endowed with the discrete topology and the binary operation "$+'$" such that $x +' y \equiv x + y (\operatorname{mod} n)$ for each $x, y \in C_n$. Let $G = \bigoplus_{n=1}^{\infty} C_n$ be the direct sum. Let $\mathcal{F}$ be the family of all non-decreasing unbounded functions from $\omega \backslash \{0\}$ to ω. For each $f \in \mathcal{F}$ put

$$O_f = \{0\} \cup \{(x_n) \in G : (\exists m \geq 1)((\forall n > m)(x_n = 0) \& (0 < x_m < f(m)))\}.$$

It is easy to check that the family $\{O_f : f \in \mathcal{F}\}$ is a base at the zero of a T_1 semigroup topology on G.

For each positive integer $m \geq 2$ we put $a^m = (a^m_n) \in G$, where $a^m_n = 1$ if $n = m$; $a^m_n = 0$ in the opposite case. Let $f, g \in \mathcal{F}$ be arbitrary functions and let $x \in G$ be an arbitrary element. There exists a number m such that $f(m) \geq 2$, $g(m) \geq 2$, and $x_n = 0$ for each $n \geq m$. Then $a^m \in O_g$ and $x + a^m \in O_f$. Therefore $x \in \overline{O_f}$. Hence $\overline{O_f} = G$ for each $f \in \mathcal{F}$. Thus G is not Hausdorff. Since each two nonempty open subsets of G intersects, we see that each two nonempty open subsets of each power of G in the box topology intersects too. Therefore each power of G in the box topology is pseudocompact. Put $A = \{0\} \cup \{a^m : m \geq 2\}$. Then A is compact and dense in G. Let κ be an arbitrary cardinal. Then A^κ is compact and dense in G^κ. Therefore the space G^κ is countably compact at a dense subset of G^κ. Thus the space G^κ is countably pracompact. □

Problem 2. *Suppose a paratopological group G satisfies one of the following conditions:*

- *G is Hausdorff countably pracompact topologically periodic left ω-precompact,*
- *G is Hausdorff countably pracompact topologically periodic,*
- *G is Hausdorff pseudocompact topologically periodic left ω-precompact,*
- *G is Hausdorff pseudocompact topologically periodic,*

Is G a topological group?

If a paratopological group G satisfies one of Problem 2 Conditions 1-4 and the group G is quasiregular or saturated or Baire then G is a topological group.

Denote by TT the following axiomatic assumption: there is an infinite torsion-free abelian countably compact topological group without non-trivial convergent sequences. The first example of such a group constructed by M. Tkachenko under the Continuum Hypothesis [Tka]. Later, the Continuum Hypothesis weakened to the Martin Axiom for σ-centered posets by Tomita in [Tom2], for countable posets in [KosTomWat], and finally to the existence continuum many incomparable selective ultrafilters in [MadTom]. Yet, the problem of the existence of a countably compact group without convergent sequences in ZFC seems to be open, see [DikSha].

The proof of [BanDimGut, Lemma 6.4] implies the following

Lemma 11. *(TT) Let G be a free abelian group generated by the set $\mathfrak{c}$. There exists a Hausdorff group topology on G such that for each countable infinite subset M of the group G there exists an element $\alpha \in \overline{M} \cap \mathfrak{c}$ such that $M \subset \langle\alpha\rangle$.* □

Example 2. *(TT) There exists a functionally Hausdorff countably compact free abelian paratopological group (G, σ) such that (G, σ) is not a topological group.*

Proof. Let G be a free abelian group generated by the set $\mathfrak{c}$. Lemma 11 implies that there exists a Hausdorff group topology τ on the group G such that for each countable infinite subset M of the group G there exists an element $\alpha \in \overline{M} \cap \mathfrak{c}$ such that $M \subset \langle\alpha\rangle$. Each non-zero element $x \in G$ has a unique representation $x = \sum_{\alpha \in A_x} n_\alpha \alpha$ such that the integer number n_α is non-zero for each $\alpha \in A_x$. Put $S_0 = \{x \in G \setminus \{0\} : n_{\sup A_x} < 0$ and $S = S_0 \cup \{0\}$. Clearly, S is a subsemigroup of the group G. It is easy to check that the family $\{U \cap S : U \in \tau, U \ni 0\}$ is a base at the zero of a semigroup topology on G. Denote this semigroup topology by σ.

Let M be an arbitrary countable infinite subset of the group G. There exists an element $\alpha \in \overline{M}^\tau \cap \mathfrak{c}$ such that $M \subset \langle\alpha\rangle$. Since $M \subseteq \alpha + S_0$, we see that $\alpha \in \overline{M}^\sigma$. Hence (G, σ) is a countably compact Hausdorff paratopological group.

Suppose (G, σ) is a topological group; then there exists a neighborhood $U \in \tau$ of the zero such that $U \cap S \subset S \cap (-S) = \{0\}$. Fix any element $\alpha \in \mathfrak{c}$. Then $-\alpha \in S$. Since the topological group

(G,τ) is countably compact, there exists a number $n<0$ such that $n\alpha\in U$. Then $0\neq n\alpha\in U\cap S$. This contradiction proves that (G,σ) is not a topological group. □

Example 2 negatively answers Problem 1 from [Gur] under TT. Since each countably compact left ω-precompact paratopological group is a topological group [Rav3, p. 82], we see that Example 2 implies the negative answers to Question A from [AlaSan] and to Problem 2 from [Gur] under TT.

Problem 3. *Is there a ZFC-example of a paratopological group G such that G is not a topological group but G satisfies one of the following conditions:*

- *G is Hausdorff countably compact,*
- *G is Hausdorff countably pracompact Baire,*
- *G is T_1 countably compact.*

If G is such a group then G is not topologically periodic, not saturated, not quasiregular, and not left ω-precompact. Moreover, $G\times G$ is not countably compact. By Theorem 2.2 from [AlaSan], a group (G,σ) from Example 2 is Baire. Therefore under TT there is a group G such that G satisfies all conditions listed in Problem 3.

The following proposition answers Questions B from [AlaSan]. Also this and the next propositions positively answer a special case for 2-pseudocompact groups of Problem 8.1 from [ArhChoKen].

Proposition 6. *Each 2-pseudocompact paratopological group of countable pseudocharacter is a topological group.*

Proof. Let G be such a group. By induction we can build a sequence $\{V_i : i\in\omega\}$ of open neighborhoods of the unit e of G such that $V_{i+1}^2\subset V_i$ for each $i\in\omega$ and $\bigcap\{V_i : i\in\omega\}=\{e\}$. Suppose G is not a topological group; then there exists a neighborhood $U\in\mathcal{B}^G$ such that $V_i\not\subset\overline{U^{-1}}\subset(U^{-1})^2$ for each $i\in\omega$. Since the group G is 2-pseudocompact, there is a point $x\in G$ such that $xW^{-1}\cap(V_i\backslash\overline{U^{-1}})\neq\emptyset$ for each $W\in\mathcal{B}^G$ and each $i>0$. Then $xW^{-1}\cap(V_i\backslash U^{-1})\neq\emptyset$ and $Wx^{-1}\cap(V_i^{-1}\backslash U)\neq\emptyset$. Therefore $x^{-1}\in\overline{V_i^{-1}}\subset V_{i-1}^{-1}$. Hence $x=e$. But $e\in U^{-1}$ and $U^{-1}\cap(V_1\backslash\overline{U^{-1}})=\emptyset$. This contradiction proves that G is a topological group. □

We shall use the following lemmas to obtain a counterpart of Proposition 6 for Hausdorff 2-pseudocompact paratopological groups.

Lemma 12. [Rez, Th. 0.5] *A compact Hausdorff semigroup with separately continouous multiplication and two-sides cancellations is a topological group.*

Lemma 13. *Let K be a compact non-empty subset of a Hausdorff semitopological group G. Then the set $G_K=\{x\in G : xK\subset K\}$ is a compact topological group.*

Proof. Clearly, G_K is a subsemigroup of the group G. If $x\in G\setminus G_K$ then there exists a point $y\in K$ such that $xy\notin K$. Since the group G is Hausdorff, the set $G\setminus K$ is open. Since the multiplication on the group G is separately continuous, there exists an open neighborhood O_x of the point x such that $O_xy\subset G\setminus K$. Thus $Ox\subset G\setminus G_K$. Therefore G_K is a closed subset of the space G. Let $y\in K$ be an arbitrary element. If $x\in G_K$ then $x\in xKy^{-1}\subset Ky^{-1}$. Therefore $G_K\subset Ky^{-1}$. Since the set Ky^{-1} is compact, the set G_K is compact too. So, G_K is a Hausdorff compact cancellative semitopological semigroup. By Lemma 12, G_K is a topological group. □

Remark 1. Lemma 13 does not necessarily hold for T_1 paratopological groups, as the following example shows. Let $G=(\mathbb{Z},+)$ be the group of integers endowed with a topology with the base $\{\{x\}\cup\{z\in\mathbb{Z} : z\geq y\} : x,y\in\mathbb{Z}\}$. It is easy to check that G is a T_1 paratopological group. Put $K=\{x\in\mathbb{Z} : x\geq 0\}$. Then K is a compact subset of the group G and $G_K=K$ is not a group.

Lemma 14. [Rav, Pr. 1.8] *Let G be a paratopological group, $K \subset G$ be a compact subspace, $F \subset G$ be a closed set, and $K \cap F = \emptyset$. Then there exists a neighborhood V of the unit such that $VK \cap F = \emptyset$.*

Proposition 7. *Each 2-pseudocompact Hausdorff paratopological group containing a compact non-empty G_δ-set is a topological group.*

Proof. Let G be such a group and K be a compact non-empty G_δ-set of the space G. Suppose G is not a topological group; then there exists a neighborhood $U \in \mathcal{B}^G$ such that $V \not\subset \overline{U^{-1}} \subset (U^{-1})^2$ for each neighborhood $V \in \mathcal{B}^G$.

By induction using Lemma 14 we can build a sequence $\{V_i : i \in \omega\}$ of open neighborhoods of the unit e of G such that $V_0 = U$, $V_{i+1}^2 \subset V_i$ for each $i \in \omega$ and $\bigcap\{V_iK : i \in \omega\} = K$. Put $H = \bigcap\{V_i : i \in \omega\}$. The construction of H implies that H is a semigroup. Moreover, since $\overline{V_{i+1}^{-1}} \subset V_{i+1}^{-1}V_{i+1}^{-1} \subset V_i^{-1}$ for each $i \in \omega$, we see that H^{-1} is a closed subset of G. Moreover, $HK \subset \bigcap\{V_iK : i \in \omega\} = K$, so $H \subset G_K$. Since G_K is a group then $H^{-1} \subset G_K$ too, so H^{-1} is a compact cancellative topological semigroup. By Lemma 12, H^{-1} is a group.

We have $V_i \not\subset \overline{U^{-1}}$ for each $i \in \omega$. Since the group G is 2-pseudocompact, there is a point $x \in G$ such that $xW^{-1} \cap (V_i \backslash \overline{U^{-1}}) \neq \emptyset$ for each $W \in \mathcal{B}^G$ and each $i > 0$. Then $xW^{-1} \cap (V_i \backslash U^{-1}) \neq \emptyset$ and $Wx^{-1} \cap (V_i^{-1} \backslash U) \neq \emptyset$. Therefore $x^{-1} \in \overline{V_i^{-1}} \subset V_{i-1}^{-1}$. Hence $x \in H^{-1}$. But $H^{-1} = H \subset U \cap U^{-1}$ and $U^{-1} \cap (V_1 \backslash \overline{U^{-1}}) = \emptyset$. This contradiction proves that G is a topological group. □

The author does not know do counterparts of Proposition 7 hold for T_1 2-pseudocompact or countably compact paratopological groups. But a T_0 sequentially compact paratopological group G_S from Example 2 from [Rav6] contains an open compact subsemigroup S. Nevertheless, G_S is not a topological group.

Example 3 is an extension of Example 5.17 from [Rav3]. Example 3 shows that counterparts of Proposition 6 and Implication ($4 \Rightarrow 1$) of Proposition 3 do not hold for countably pracompact paratopological groups. Also Example 3 negatively answers Problem 6.2 from [Tka2] and Problem 8.1 from [ArhChoKen].

Example 3. *There exists a functionally Hausdorff second countable paratopological group G such that each power of G is countably pracompact but G is not a topological group.*

Proof. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Denote by τ the standard topology on the group $\mathbb{T}$. For each $\varepsilon > 0$ put $B_\varepsilon(0_\mathbb{T}) = \{z \in \mathbb{T} : |\arg z| < \varepsilon\}$. Let $s : \mathbb{T} \to \mathbb{Q}$ be a homomorphism onto. For each $a \in \mathbb{R}$ put $\mathbb{T}_a = \{x \in \mathbb{T} : s(x) > a\}$ and $\mathbb{T}_a^0 = \{x \in \mathbb{T} : s(x) \geq a\}$. Put $\mathcal{B}_1' = \{\mathbb{T}_a : a < 0\}$, $\mathcal{B}_2' = \{\mathbb{T}_0^0\}$, $\mathcal{B}_3' = \{\{0_\mathbb{T}\} \cup \mathbb{T}_0\}$ and $\mathcal{B}_4' = \{\{0_\mathbb{T}\} \cup \mathbb{T}_a : a > 0\}$. It is easy to check that $\mathcal{B}_i'$ is a base at the unit of a first countable semigroup topology on the group $\mathbb{T}$ for every i. Therefore for every i the family $\mathcal{B}_i = \{S \cap B_\varepsilon(0_\mathbb{T}) : S \in \mathcal{B}_i', \varepsilon > 0\}$ is a base at the unit of a first countable functionally Hausdorff semigroup topology on the group $\mathbb{T}$. Denote this topology by τ_i and put $G_i = (\mathbb{T}, \tau_i)$. Clearly, $\tau \subset \tau_1 \subset \tau_2 \subset \tau_3 \subset \tau_4$. Choose an element $x_0 \in \mathbb{T}_0$. It is easy to check that the set $A = \{nx_0 : n \in \mathbb{Z}, n \geq 0\}$ is dense in G_4. Let κ be an arbitrary cardinal. Then A^κ is dense in G_4^κ. Let $\{y^n : n \in \omega\}$ be an arbitrary sequence of points from A^κ. Since the space $(\mathbb{T}, \tau)^\kappa$ is compact, we see that there is a point $y \in \mathbb{T}^\kappa$ such that the set $\{n \in \omega : y^n \in U\}$ is infinite for each open in $(\mathbb{T}, \tau)^\kappa$ set $U \ni y$.

We claim that each open in G_4^κ set $U \ni y$ has a similar property. Fix such a neighborhood U. For each $\alpha \in \kappa$ let $\pi_\alpha : \mathbb{T}^\kappa \to \mathbb{T}$ be the projection onto the α-th coordinate. There exist a finite subset F of κ, a family $\{U_\alpha : \alpha \in F\}$ of open in $(\mathbb{T}, \tau)$ sets, and a number $a > 0$ such that $V' \subset U$, where $V' = \bigcap_{\alpha \in F} \pi_\alpha^{-1}((y_\alpha + \mathbb{T}_a) \cap U_\alpha)$. Decreasing neighborhoods U_α, if necessary, we can assume that for each $\alpha \in F$ and for every integer $0 < n \leq (s(y_\alpha) + a)/s(x_0)$ if $nx_0 \neq y_\alpha$, then $nx_0 \notin U_\alpha$.

Put $V=\bigcap_{\alpha\in F}\pi^{-1}(U_\alpha)$. Now let z be an arbitrary point of $A^\kappa\cap V$. Then $z_\alpha=y_\alpha$ or $s(z_\alpha)>s(y_\alpha)$ for each $\alpha\in F$. Therefore $z\in V'$. Hence the set $\{n\in\omega: y^n\in U\}\supset\{n\in\omega: y^n\in V'\}=\{n\in\omega: y^n\in V\}$ is infinite. Therefore the space G_4^κ is countably compact at its dense subset A^κ. Thus the space G_4^κ is countably pracompact.

Let $i=1$ or $i=2$. A set $s^{-1}(0)$ is a subgroup of $\mathbb{T}$ of countable index and $\tau_i|s^{-1}(0)=\tau|s^{-1}(0)$. Since the group $(\mathbb{T},\tau)$ has the countable weight, the group G_i has a countable network. Since $\chi(G_i)=\omega$, by [Rav, Pr. 2.3], the group G_i is second countable.

We claim that the group G_2 has no countable network of closed subsets of G_2. Assume the converse. Let $\mathcal{N}$ be a countable network such that for each set $X\in\mathcal{N}$ there exists an element $x\in G$ with $x+\mathbb{T}_0^0\supset X$. We claim that the set X is nowhere dense in (G,τ). Assume the converse. Since the set $\{nx_0: n\in\mathbb{Z}, n<0\}$ is dense in (G,τ), there exists an element $y=nx_0\in\operatorname{int}_\tau\overline{X}^\tau$ such that $s(y)<s(x)$. Let $\varepsilon>0$ be an arbitrary number such that $y+B_\varepsilon(0_\mathbb{T})\subset\overline{X}^\tau$. There exists a point $z\in(y+B_\varepsilon(0_\mathbb{T}))\cap X$. Since $z\in X\subset x+\mathbb{T}_0^0$, we see that $s(z)\ge s(x)>s(y)$. Thus $z\in y+\mathbb{T}_0^0$. Hence $z\in y+(B_\varepsilon(0_\mathbb{T})\cap\mathbb{T}_0^0)$. Therefore $y\in\overline{X}^{\tau_2}=X\subset x+\mathbb{T}_0^0$. Thus $s(y)\ge s(x)$. This contradiction proves that the set X is nowhere dense in (G,τ). Then $(G,\tau)=\bigcup\mathcal{N}$ is a union of a countable family of its closed nowhere dense sets. This contradiction proves that the group G_2 has no countable network of closed subsets of G_2.

The set $\mathbb{T}\backslash\mathbb{T}_0$ is a closed discrete subset of G_4. Thus $e(G_4)=\mathfrak{c}$. The set $\{(x,-x): x\in G\}$ is a closed discrete subset of $G_3\times G_3$. Thus $e(G_3\times G_3)=\mathfrak{c}$.

Extent and Lindelöf number of the group G_3 depend on properties of the homomorphism s. The following two examples belong to T. Banakh. We delay their constructions to Section 4.

Example 4. *There is a homomorphism $s:(\mathbb{T},\tau)\to\mathbb{Q}$ onto such that the set $s^{-1}(0)$ contains a compact subset K of cardinality $\mathfrak{c}$.*

Example 5. *There is a homomorphism $s:(\mathbb{T},\tau)\to\mathbb{Q}$ onto such that for each $a\in\mathbb{Q}$ each closed subset of the set $\mathbb{T}_a$ is countable.*

Suppose that the homomorphism s satisfies the conditions of Example 4. Since $\tau\subset\tau_3$, K is a closed subset of G_3. Since $(x+\mathbb{T}_0)\cap K=\{x\}$ for each $x\in K$, K is discrete. Thus $\mathfrak{c}\le e(G_3)\le l(G_3)=\mathfrak{c}$.

Suppose that the homomorphism s satisfies the conditions of Example 5. Let $\mathcal{U}$ be an open cover of the group G_3. For each point $x\in G_3$ choose a neighborhood $O_x\in\tau$ of x and an element $U_x\in\mathcal{U}$ such that such that $(\{x\}\cup(x+\mathbb{T}_0))\cap O_x\subset U_x$. For each $a\in\mathbb{Q}$ put $A_a=\bigcup\{Ox: x\in\mathbb{T}\backslash\mathbb{T}_a\}$ and $B_a=\mathbb{T}\backslash A_a$. Then B_a is closed in $(\mathbb{T},\tau)$ subset of $\mathbb{T}_a$ and hence B_a is countable. Since $(\mathbb{T},\tau)$ is hereditarily Lindelöf there is a countable subset C_a of $\mathbb{T}\backslash\mathbb{T}_a$ such that $A_a=\bigcup\{Ox: x\in C_a\}$. Put $\mathcal{U}_a=\{U_x: x\in B_a\cup C_a\}$. Let $y\in\mathbb{T}_a$. If $y\in B_a$ then $y\in U_y\in\mathcal{U}_a$. If $y\notin B_a$ then there is a point $x\in C_a$ such that $y\in O_x$. Since $y\in\mathbb{T}_a$ and $x\in\mathbb{T}\backslash\mathbb{T}_a$ we see that $y\in(x+\mathbb{T}_0)\cap O_x\subset U_x\in\mathcal{U}_a$. Hence $\mathbb{T}_a\subset\bigcup\mathcal{U}_a$ and therefore $\mathbb{T}=\bigcup\{\mathbb{T}_a: a\in\mathbb{Q}\}\subset\bigcup\{\bigcup\mathcal{U}_a: a\in\mathbb{Q}\}$. Therefore $l(G_3)\le\omega$. □

Moreover, Manuel Sanchis and Mikhail Tkachenko constructed a Hausdorff 2-pseudocompact Fréchet-Urysohn paratopological group which is not a topological group [SanTka, Th.1], answering an author's question from a previous version of the manuscript.

Oleg Gutik asked the following question. Let G be a countably compact paratopological group and H be a subgroup of G. Is $\overline{H}$ a group? The next proposition exclaims the situation.

Proposition 8. *Let G be a countably compact paratopological group such that the closure of each cyclic subgroup of G is a group. Then G is a topological group.*

Proof. Let x be an arbitrary point of G. Put $H=\overline{\langle x\rangle}$. Then H is a separable countably compact paratopological group. Since each countably compact left ω-precompact paratopological group is a

topological group [Rav3, p. 82], we see that H is a topological group. Therefore H is a topologically periodic group. Hence G is a topologically periodic group. By [BokGur], H is a topological group. □

The following proposition answers a question of Igor Guran.

Proposition 9. *Each Hausdorff σ-compact pseudocompact paratopological group is a compact topological group.*

Proof. Let (G,τ) be such a group and let (G,τ_r) be the regularization of the group (G,τ). Then (G,τ_r) is a regular pseudocompact topological group. Let $G=\bigcup\{K_n : n\in\omega\}$ be a union such that the set K_n is compact in (G,τ) for each $n\in\omega$. Then for each $n\in\omega$ the set K_n is compact in (G,τ_r). Since the group (G,τ_r) is Baire there exists a number $n\in\omega$ such that the set K_n has nonempty interior in (G,τ_r). Since $\tau_r\subset\tau$ then the set K_n has nonempty interior in (G,τ). Therefore the group (G,τ) is locally compact. By [Rav3, Pr.5.5], (G,τ) is a topological group. Since (G,τ) is precompact, we see that (G,τ) is compact. □

4. The Constructions of Examples 4 and 5

Example 4. By [Kech, 19.2] there is a Cantor set $C\subset\mathbb{R}$ whose members are linearly independent over $\mathbb{Q}$. Pick arbitrary distinct numbers $a,b\in C$. Let $B'\supset C$ be a maximal linearly independent over $\mathbb{Q}$ subset of $\mathbb{R}$ (which exists by Zorn lemma). Then B' is a basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$. Let $s':\mathbb{R}\to\mathbb{Q}$ be a homomorphism map of vector spaces over $\mathbb{Q}$ such that $s'(a)=1$ and $s'(B'\backslash\{a\})=\{0\}$. Let $\pi:\mathbb{R}\to\mathbb{T}$ be the homomorphism such that $\pi(x)=e^{2\pi xi/b}$ for each $x\in\mathbb{R}$. It is easy to check that there exists a unique map $s:\mathbb{T}\to\mathbb{Q}$ such that $s'=s\pi$. Moreover, s is a homomorphism and $\ker s\supset\pi(B'\backslash\{a\})$. There exists an uncountable compact set $B\subset C\backslash\{a\}$. Since s is continuous and $\pi|C$ is injective, $\pi(B)$ is an uncountable compact subset of $\ker s$. By [Kech, 3.3.2 and 6.5], $|\pi(B)|=\mathfrak{c}$. □

Example 5. We shall proceed similarly to the construction of a Bernstein set. It is easy to check that $\mathbb{T}$ contains exactly $\mathfrak{c}$ uncountable closed subsets. Let $\{K_\alpha:\alpha<\mathfrak{c}\}$ be an enumeration of all uncountable closed subsets of $\mathbb{T}$. By [Kech, 3.3.2 and 6.5], each set K_α has cardinality continuum.

For a set $B\subset\mathbb{T}$ by $[B]$ we shall denote a set $\{x\in\mathbb{T} : (\exists k\in\mathbb{Z}\setminus\{0\}) : kx\in B\}$. It is easy to check that $|[B]|\le\aleph_0|B|$.

Now we shall construct the homomorphism s by transfinite recursion. For this purpose we construct a non-decreasing transfinite sequence $\{B_\alpha:\alpha<\mathfrak{c}\}$ of subgroups of $\mathbb{T}$. Let $\alpha<\mathfrak{c}$ be an ordinal. Suppose that we already have constructed the subgroups B_β (such that $B_\beta=[B_\beta]$ and $|B_\beta|\le\aleph_0\beta$) and the restrictions $s|B_\beta$ for all $\beta<\alpha$. (To assure the recursion base we formally put $B_{-1}=[\{0\}]$ and $s(B_{-1})=\{0\}$). Put $C_\alpha=\bigcup\{B_\beta:\beta<\alpha\}$. Then $|C_\alpha|\le\aleph_0\alpha<\mathfrak{c}=|K_\alpha|$. Therefore we can inductively construct a sequence $\{c_n:n\in\omega\}$ of points of K_α such that $c_n\notin C_{\alpha,n}$ for each n, where $C_{\alpha,0}=C_\alpha$ and $C_{\alpha,n+1}=[C_{\alpha,n}\cup\{c_n\}]$ for each n. Put $B_\alpha=\bigcup\{C_{\alpha,n}:n\in\omega\}$. Then $|B_\alpha|\le\aleph_0\alpha$. We define the restriction $s|B_\alpha$ inductively. The restriction $s|C_{\alpha,0}$ is already defined. Suppose that we already defined the restriction $s|C_{\alpha,n}$. Let x be an element of $C_{\alpha,n+1}$. Then there exist integers $k\ne0$ and l such that $kx=c+lc_n$, where $c\in C_{\alpha,n}$. Put $s(x)=(s(c)-ln)/k$. Now we have to show that this definition is correct, that is it does not depend on the integers k and l. Suppose that $k'x=c'+l'c_n$, for element $c'\in C_{\alpha,n}$ and integers $k'\ne0$ and l'. Then $(lk'-l'k)c_n=kc'-k'c\in C_{\alpha,n}$. Since $c_n\notin C_{\alpha,n}=[C_{\alpha,n}]$, we see that $lk'=l'k$ and thus $kc'=k'c$. Then $(s(c)-ln)/k=(s(c')-l'n)/k'$. In particular, we have that $(s|C_{\alpha,n+1})|C_{\alpha,n}=s|C_{\alpha,n}$. It is easy to check that $s|C_{\alpha,n+1}$ is a homomorphism. Then $s|B_\alpha$ is a homomorphism too and $K_\alpha\not\subset\mathbb{T}_a$ for each $a\in\mathbb{Q}$ because $\{c_n\}\subset K_\alpha$ and $s(c_n)=-n$ for each n. The construction implies that $s|B$ is a homomorphism and $\inf s|B(K)=-\infty$ for each uncountable closed subset K of $\mathbb{T}$. Since B is a

divisible subgroup of an abelian group $\mathbb{T}$, there exists an extension s of a homomorphism $s|B$ onto $\mathbb{T}$ (see, for instance, [Kur, § 23]). $\square$

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